Homework 4
Algebra

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Proposition 0.1 (Exercise 1)**.** *Every abelian group of order 1008 is, up to isomorphism, a direct sum from one of each column:*

$A(2)$	$A(3)$	$A(7)$
$\mathbb{Z}_{16}$	$\mathbb{Z}_9$	$\mathbb{Z}_7$
$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	
$\mathbb{Z}_4 \oplus \mathbb{Z}_4$		
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$		
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$		

Proof. Let A be an abelian group of order 1008. Note that $1008 = 2^4 3^2 7$. By Theorem 8.1 (Lang), we have

$$A = A(2) \oplus A(3) \oplus A(7)$$

and by Theorem 8.2 (Lang), $A(2)$ is isomorphic to one of $\mathbb{Z}_16, \mathbb{Z}_8 \oplus \mathbb{Z}_2, \mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By the same theorem, $A(3)$ is isomorphic to one of $\mathbb{Z}_9, \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and $A(7) \cong \mathbb{Z}_7$. Thus we can take any combination of these, and each choice is unique up to isomorphism, as $2, 3, 7$ are pairwise relatively prime, and reordering a direct product does not change the isomorphism class. □

Proposition 0.2 (Exercise 2a)**.** *Let $\{A_i\}_{i \in I}$ be a family of abelian groups of exponent m. Then*

$$\left(\bigoplus_{i \in I} A_i \right)^\wedge \cong \prod_{i \in I} (A_i^\wedge)$$

Proof. Let $\lambda_i : A_i \to \oplus_i A_i$ be the map $x \mapsto (0, \ldots, x, \ldots 0)$. We define two maps,

$$\left(\bigoplus_{i \in I} A_i \right)^\wedge \to \prod_{i \in I} (A_i^\wedge) \qquad \prod_{i \in I} (A_i^\wedge) \to \left(\bigoplus_{i \in I} A_i \right)^\wedge$$
$$\psi \mapsto (\psi \circ \lambda_i) \qquad (\psi_i) \mapsto \psi$$

On the right, ψ is the unique homomorphism from $\oplus A_i$ to $\mathbb{Z}/m\mathbb{Z}$ such that $\psi \circ \lambda_i = \psi_i$ for all i (this homomorphism exists and is unique by the universal property of the direct sum, Lang

Proposition 7.1). First we show that both maps are homomorphisms. Let $\psi, \chi \in (\oplus A_i)^\wedge$. Then

$$\psi + \chi \mapsto ((\psi + \chi) \circ \lambda_i) = (\psi \circ \lambda_i + \chi \circ \lambda_i) = (\psi \circ \lambda_i) + (\chi \circ \lambda_i)$$

so the left map is a homomorphism. Now let $(\psi_i), (\chi_i) \in \prod(A_i^\wedge)$, and let ψ, χ be their respective images in $(\oplus A_i)^\wedge$.

$$(\psi_i) + (\chi_i) = (\psi_i + \chi_i) \mapsto \phi$$

where $\phi \circ \lambda_i = \psi_i + \chi_i$. But we also have

$$(\psi + \chi) \circ \lambda_i = \psi \circ \lambda_i + \chi \circ \lambda_i = \psi_i + \chi_i$$

so by the uniqueness of the map ϕ, we have $\phi = \psi + \chi$, thus $(\psi_i) + (\chi_i) \mapsto \psi + \chi$, so the right map is also a homomorphism. Now we claim that these maps are inverse to each other. In one direction, for any $(\psi_i) \in \prod(A_i^\wedge)$ we have the composition

$$(\psi_i) \mapsto \psi \mapsto (\psi \circ \lambda_i) = (\psi_i)$$

so we have the identity on $\prod(A_i^\wedge)$. In the other direction, for any $\psi \in (\oplus A_i)^\wedge$, we have

$$\psi \mapsto (\psi \circ \lambda_i) \mapsto \phi$$

where ϕ is the unique homomorphism such that $\phi \circ \lambda_i = \psi \circ \lambda_i$. Obviously, ψ is also a homomorphism satisfying $\psi \circ \lambda_i = \psi \circ \lambda_i$, so $\phi = \psi$. Thus this composition is the map $\psi \mapsto \psi$, so it is the identity on $(\oplus A_i)^\wedge$. Thus these maps are both isomorphisms. □

Proposition 0.3 (Exercise 2b)**.** *It is not true in general that*

$$\left(\prod_{i \in I} A_i\right)^\wedge \cong \bigoplus_{i \in I} A_i^\wedge$$

Proof. Let $A_i = \mathbb{Z}/2\mathbb{Z}$ and let $I = \mathbb{N}$. Then $A_i^\wedge \cong \mathbb{Z}/2\mathbb{Z}$, so we have

$$\bigoplus_{i \in I} A_i^\wedge = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

which is countable. On the other hand,

$$\prod_{i \in I} A_i = \prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

is uncountable, since it can be put in one-to-one correspondence with the real numbers between 0 and 1, if we think of each element as a binary expansion. More importantly, we claim that

$$\left(\prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}\right)^\wedge = \text{Hom}\left(\prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\right)$$

is uncountable. For each nonzero element $x = (x_i) \in \prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, we define a map $\phi_x : \prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ such that $\phi_x(x) = 1$ and $\phi_x(y) = 0$ for $y \neq x$. In terms of the Kronecker delta function, we can write this as $\phi_x(y) = \delta_x^y$. This is a homomorphism, as

$$\begin{aligned} \phi_x(y+z) = \delta_x^{y+z} &= \begin{cases} 0 & y, z \neq x \text{ or } y = z = x \\ 1 & y = x \text{ or } z = x \text{ but not both} \end{cases} \\ &= \delta_x^y + \delta_x^z = \phi_x(y) + \phi_x(z) \end{aligned}$$

And each ϕ_x is a different homomorphism from each other ϕ_x. Thus there are uncountably many distinct elements of the dual of the direct product, so there cannot be a bijection. Thus they are not isomorphic. □

Proposition 0.4 (Exercise 2c)**.** *There is an abelian group A of exponent m such that A is not isomorphic to $A^\wedge$ or to $A^{\wedge\wedge}$.*

Proof. Let $A = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/m\mathbb{Z}$. Since $\mathbb{Z}/m\mathbb{Z}$ is finite, $\mathbb{Z}/m\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z})^\wedge$, so by part (a),

$$A^\wedge = \left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/m\mathbb{Z} \right)^\wedge \cong \prod_{n\in\mathbb{N}} \mathbb{Z}/m\mathbb{Z}$$

But as shown in part (b), this infinite direct sum is not isomorphic to the infinite direct product, so A cannot be isomorphic to $A^\wedge$. By a similar argument as in part (b), the dual of the uncountable direct product is also uncountable, so A is not isomorphic to $A^{\wedge\wedge}$. □

Proposition 0.5 (Exercise 3a)**.** *Let I be a partially ordered indexing set, and let $\{G_i\}_{i\in I}$ be an inversely directed family of groups and $f_i^j : G_j \to G_i$ for $i \leq j$ and let*

$$\Gamma = \varprojlim G_i$$

Let $G = \prod_{i\in I} G_i$ and let $p_i : G \to G_i$ be the canonical projection $(x_i) \mapsto x_i$. Let $\iota : \Gamma \to G$ be the inclusion map. Then define $\pi_i : \Gamma \to G_i$ by $\pi_i = \iota \circ p_i$. Then π_i is a canonical projection homomorphism.

Proof. Γ is a subgroup of G as noted in Lang, thus the inclusion is a homomorphism, i.e. $\iota(xy) = xy) = \iota(x)\iota(y)$. The projection $p_i : G \to G_i$ is also a homomorphism, because if we have $x = (x_i) \in G$ and $y = (y_i) \in G$, then

$$p_i(xy) = (xy)_i = x_i y_i = p_i(x)p_i(y)$$

Thus π_i is a composition of homomorphisms so it is a homomorphism. □

Proposition 0.6 (Exercise 3b)**.** *Let $\Gamma = \varinjlim G_i$. Then let H be a group and $\phi_i : H \to G_i$ be a family of homomorphisms such that $f_i^j \circ \phi_i$ whenever $i \leq j$. Then define $\phi : G \to \Gamma$ by $h \mapsto (\phi_i(h))$. Then ϕ is the unique homomorphism from H to Γ such that $\phi_i = \pi_i \circ \phi$ for all i.*

Proof. First, we show that ϕ maps into Γ. To do this, we need to show that $f_i^j \circ \pi_j \circ \phi(h) = \pi_i \circ \phi(h)$. We have $\pi_i \circ \phi = \phi_i$, so

$$f_i^j \circ \pi_j \circ \phi(h) = f_i^j \circ \phi_j(h) = \phi_i(h) = \pi_i \circ \phi(h)$$

Thus ϕ maps into Γ. Now we show that ϕ is a homomorphism.

$$\phi(hk) = (\phi_i(hk)) = (\phi_i(h)\phi_i(k)) = (\phi_i(h))(\phi_i(k)) = \phi(h)\phi(k)$$

Thus ϕ is a homomorphism. It is clear that $\phi_i \circ \pi_i \circ \phi$. Finally, we need to show that ϕ is unique. Suppose $\psi : H \to \Gamma$ is also a homomorphism such that $\pi_i \circ \psi = \phi_i$. Then

$$\pi_i \circ \psi(h) = \phi_i(h) \implies \psi(h) = (\phi_i(h))$$

thus $\psi = \phi$. Thus ϕ is unique. □

Lemma 0.7 (for Exercise 4)**.** *Let G be a finitely generated abelian group and H a subgroup. Then G/H is a finitely generated abelian group.*

Proof. Let $\{x_1, \dots x_n\}$ be a generating set for G and let $gH \in G/H$. Then g can be written as $g = \sum_i a_i x_i$ where $a_i \in \mathbb{Z}$. Then $gH = (\sum_i a_i x_i)H = \sum_i a_i(x_i H)$ so $\{x_i H\}$ is a generating set for G/H. □

Lemma 0.8 (for Exercise 4)**.** *Let G be a finitely generated abelian group of rank n and H a subgroup of rank m. Then G/H_{tor} is a finitely generated abelian group of rank n.*

Proof. G/H_{tor} is a finitely generated abelian group by the previous lemma. Let $\{x_1, \dots x_n\}$ be a basis for G_{free}. Let $\pi : G \to G/H_{\text{tor}}$ be the canonical projection. We claim that $\{\pi(x_1), \dots \pi(x_n)\}$ is a basis for the free part of G/H_{tor}. Let $gH_{\text{tor}} \in (G/H_{\text{tor}})_{\text{free}}$. Then $g \in G$ so g can be written uniquely as $g = y + \sum_{i=1}^n a_i x_i$ where y has finite order. Then

$$gH_{\text{tor}} = \pi\left(y + \sum_{i=1}^n a_i x_i\right) = \pi(y) + \sum_{i=1}^n a_i \pi(x_i) = \sum_{i=1}^n a_i \pi(x_i)$$

Since y has finite order, so does $\pi(y)$; therefore, because $g \in (G/H_{\text{tor}})_{\text{free}}$, we must have $\pi(y) = 0$. We now need to show that this linear combination is unique. If we have

$$gH_{\text{tor}} = \sum_{i=1}^n a_i \pi(x_i) = \sum_{i=1}^n b_i \pi(x_i)$$

Then we must have

$$H_{\text{tor}} = \sum_{i=1}^n (b_i - a_i)\pi(x_i) = \left(\sum_{i=1}^n (b_i - a_i)x_i\right) H_{\text{tor}} \implies \sum_{i=1}^n (b_i - a_i)x_i \in H_{\text{tor}}$$

But every non-zero linear combination of $\{x_i\}$ lies in G_{free}, so it has infinite order. Therefore the only linear combination of $\{x_i\}$ in H_{tor} is the trivial one, so $b_i = a_i$. Thus the linear combination is unique. □

Proposition 0.9 (Exercise 4). *Let G be a finitely generated abelian group of rank n. Let H be a subgroup of rank m. Then G/H has rank $n-m$, and in particular, $m \leq n$.*

Proof. By the third isomorphism theorem,

$$G/H \cong \frac{G/H_{\text{tor}}}{H/H_{\text{tor}}}$$

As shown above, G/H_{tor} is a finitely generated abelian group of rank n. H/H_{tor} is free of rank m by Theorem 8.5 (Lang), thus H/H_{tor} must be contained in the free part of G/H_{tor}, so we have

$$\frac{G/H_{\text{tor}}}{H/H_{\text{tor}}} \cong \frac{(G/H_{\text{tor}})_{\text{free}}}{H/H_{\text{tor}}} \oplus (G/H_{\text{tor}})_{\text{tor}}$$

By the above lemma, $(G/H_{\text{tor}})_{\text{free}} \cong \mathbb{Z}^n$, so

$$\frac{(G/H_{\text{tor}})_{\text{free}}}{H/H_{\text{tor}}} \oplus (G/H_{\text{tor}})_{\text{tor}} \cong \frac{\mathbb{Z}^n}{\mathbb{Z}^m} \oplus (G/H_{\text{tor}})_{\text{tor}} \cong \mathbb{Z}^{n-m} \oplus (G/H_{\text{tor}})_{\text{tor}}$$

Hence

$$G/H \cong \mathbb{Z}^{n-m} \oplus (G/H_{\text{tor}})_{\text{tor}}$$

so G/H has rank $n-m$. As finitely generated abelian groups have non-negative rank, this implies that $m \leq n$. □

Proposition 0.10 (Exercise 24). *Let p be a prime, and let G be a group of order p^2. Then G is abelian, and there are only two such groups up to isomorphism.*

Proof. We know that $Z(G)$ is nontrivial. By Lagrange's Theorem, it must have order p^2 or p. If it has order p^2, then $Z(G) = G$ so G is abelian. Suppose that the order of $Z(G)$ is p. We know the center is a normal subgroup, so $|G/Z(G)| = |G|/|Z(G)| = p^2/p = p$. Thus $G/Z(G)$ has order p, so it is cyclic. Then by a previous result, G is abelian.

Now we show there are precisely two groups of order p^2. By Theorem 8.2 (Lang), G is isomorphic to either $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$. □

Proposition 0.11 (Exercise 43). *Let G be a finite abelian group and H a subgroup. Then G has a subgroup isomorphic to G/H.*

Proof. By the classification of finite abelian groups, we can write G as

$$G \cong \bigoplus_{i=1}^{m} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

where the p_i may not all be distinct. Then a subgroup H must be isomorphic to

$$H \cong \bigoplus_{i=1}^{m} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$$

where $k_i \leq n_i$. Then it follows that

$$G/H \cong \bigoplus_{i=1}^{m} \mathbb{Z}/p_i^{n_i-k_i}\mathbb{Z}$$

which is isomorphic to a subgroup of G. □

Lemma 0.12 (for Exercise 50a). *Let X_i be a family of abelian groups and let A be an abelian group and let $\phi_i : X_i \to A$ be a family of group homomorphisms. Define*

$$X = \left\{ (x_i)_{i\in I} \in \bigoplus_{i\in I} X_i \,\middle|\, \phi_j(x_j) = \phi_k(x_k) \text{ for all } j,k \in I \right\}$$

Then X is a subgroup of $\bigoplus_{i\in I} X_i$.

Proof. We know that X contains the identity $(0,0,\ldots)$ as we have $\phi_j(0) = \phi_k(0) = 0$ for all $j,k \in I$. If $(x_i)_{i\in I}, (y_i)_{i\in I} \in X$, then their sum is $(x_i + y_i)$, and so

$$\phi_j(x_j + y_j) = \phi_j(x_j) + \phi_j(y_j) = \phi_k(x_k) + \phi_k(y_k) = \phi_k(x_k + y_k)$$

so $(x_i) + (y_i) \in X$. The inverse of $(x_i)_{i\in I}$ is $(-x_i)_{i\in I}$, as

$$(x_i) + (-x_i) = (x_i - x_i) = (0, \ldots)$$

□

Proposition 0.13 (Exercise 50a). *Fiber products exist in the category of abelian groups. In particular, in the case of just two group homomorphisms $\phi_1 : X_1 \to A$ and $\phi_2 : X_2 \to A$, we have*

$$X_1 \times_A X_2 = \{(x_1, x_2) \in X_1 \oplus X_2 : \phi_1(x_1) = \phi_2(x_2)\}$$

Proof. Let $\mathbf{Ab}$ be the category of abelian groups. Let A be an abelian group and let $\mathbf{Ab}_A$ be the category of morphisms into A. We need to show that products exist in $\mathbf{Ab}_A$. More precisely, we need to show that for every family $\{\phi_i\}_{i\in I}$ of objects in $\mathbf{Ab}_A$, there exists $\phi \in \mathrm{Ob}(\mathbf{Ab}_A)$ and a family of morphisms $p_i \in \mathrm{Mor}(\phi, \phi_i)$ such that for every $\psi \in \mathrm{Ob}(\mathbf{Ab}_A)$ with a family of morphisms $g_i \in \mathrm{Mor}(\psi, \phi_i)$ there is a unique morphism $h \in \mathrm{Mor}(\psi, \phi)$ such that $p_i \circ h = g_i$ for all i.

Let $\{\phi_i\}_{i\in I}$ be a family of objects in $\mathbf{Ab}_A$, that is, let $\phi_i : X_i \to A$ be a family of group homomorphisms from abelian groups X_i into A. Let

$$X = \left\{ (x_i) \in \bigoplus_{i\in I} X_i \,\middle|\, \phi_j(x_j) = \phi_k(x_k) \text{ for all } j,k \in I \right\}$$

As shown above, X is a subgroup of $\bigoplus_{i\in I} X_i$, so X is an abelian group. Then we define $\phi : X \to A$ by $(x_i)_{i\in I} \mapsto \phi_i(x_i)$ where i can be any $i \in I$. This is well-defined as $\phi_i(x_i) = \phi_j(x_j)$ for any $j \in I$. Note that ϕ is a homomorphism, because if we have $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ with $x, y \in X$, then

$$\phi(x + y) = \phi((x_i + y_i)_{i\in I}) = \phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i) = \phi(x) + \phi(y)$$

Thus ϕ is an object in $\mathbf{Ab}_A$. Let $p_i : X \to X_i$ be the projection $(x_j)_{j\in J} \mapsto x_i$. Also, p_i is a morphism in $\mathbf{Ab}_A$, because $\phi_i \circ p_i = \phi$. In particular, $p_i \in \mathrm{Mor}(\phi, \phi_i)$.

$$\phi_i \circ p_i(x) = \phi_i(x_i) = \phi(x)$$

We claim that $(\phi, \{p_i\})$ is a product in $\mathbf{Ab}_A$. Let ψ be an object in $\mathbf{Ab}_A$ and let $g_i \in \mathrm{Mor}(\psi, \phi_i)$. Then ψ is a group homomorphism from some abelian group Y into A and each g_i is a group homomorphism from Y to X_i such that $\psi = \phi_i \circ g_i$. We need $h \in \mathrm{Mor}(\psi, \phi)$ such that $p_i \circ h = g_i$ for all i. That is, we need a group homomorphism $h : Y \to X$ such that $h \circ p_i = g_i$. Define $h : Y \to X$ by

$$y \mapsto (g_i(y))_{i \in I}$$

Then $\phi \circ h(y) = \phi((g_i(y)) = \phi_i(g_i(y)) = \psi(y)$ so $h \in \mathrm{Mor}(\psi, \phi)$. Furthermore, $p_i \circ h(y) = g_i(y)$ so we have the needed $p_i \circ h = g_i$. Finally, we just need to show that h is unique. If $f : Y \to X$ is another homomorphism such that $p_i \circ f = g_i$, then $f(y) = (g_i(y))_{i \in I}$, so $f = h$. Thus h is unique. □

Proposition 0.14 (Exercise 50b)**.** *In the category of abelian groups, the pull-back of a surjective homomorphism is surjective.*

Proof. Let X, Y, Z be abelian groups and let $f : X \to Z$ and $g : Y \to Z$ be group homomorphisms, and let $X \times_Z Y$ be the fiber product. Let $p_2 : X \times_Z Y \to Y$ be the pull-back of f by g. As shown in part (a), $p_2(x, y) = y$. We suppose that f is surjective, and want to show that p_2 is surjective. Let $y \in Y$. Then $g(y) \in X$, so by surjectivity of f there exists $x \in X$ such that $f(x) = g(y)$. Thus $(x, y) \in X \times_Z Y$ so $p_2(x, y) = y$. Thus p_2 is surjective. □

Proposition 0.15 (Exercise 52a)**.** *Fiber coproducts exist in the category of abelian groups. In particular, the fiber coproduct of two homomorphisms f, g is the factor group $X \oplus_Z Y = (X \oplus Y)/W$ where $W = \{(f(z), -g(z)) \in X \oplus Y : z \in Z\}$.*

Proof. Let f, g be objects in $\mathbf{Ab}^Z$, so X, Y are abelian groups and $f : Z \to X$ and $g : Z \to Y$ are group homomorphisms. Let W and $X \oplus_Z Y$ be as described. Define $i_1 : X \to X \oplus_Z Y$ and $i_2 : Y \to Y \oplus_Z Y$ by

$$i_1(x) = (x, 0)W \qquad i_2(y) = (0, y)W$$

and define $\phi : Z \to X \oplus_Z Y$ by $z \mapsto (f(z), 0)W$. Then we have

$$\phi(z) = i_1 \circ f(z) = (f(z), 0)W = (f(z), 0)W - (f(z), -g(z))W = (0, g(z))W = i_2 \circ g(z)$$

We claim that $(\phi, \{i_1, i_2\})$ is a fiber coproduct of f and g. Let $j_1 : Y \to C$ and $j_2 : X \to C$ and $\psi : Z \to C$ be group homomorphisms such that $j_1 \circ f = \psi$ and $j_2 \circ g = \psi$, that is, $j_1 \in \mathrm{Mor}(f, \psi)$ and $j_2 \in \mathrm{Mor}(g, \psi)$. Then define $u : X \oplus_Z Y \to C$ by

$$(x, y)W \mapsto j_1(x) + j_2(y)$$

Then u is well-defined because if $(x, y)W = (x', y')W$ then we have $f(z) = x - x'$ and $-g(z) = y - y'$ for some $z \in Z$, so

$$\begin{aligned} u((x, y)W) - u((x', y')W) &= j_1(x) + j_2(y) - j_1(x') - j_2(y') \\ &= j_1(x - x') + j_2(y - y') = j_1 \circ f(z) - j_2 \circ g(z) = 0 \end{aligned}$$

Note that u is a homomorphism because j_1, j_2 are homomorphisms. Then we have $j_1 \circ f = u \circ i_1 \circ f$ easily, and for the other required commutative diagram we check that $u \circ i_2 \circ g = j_2 \circ g$.

$$u \circ i_2 \circ g(z) = u((0, g(z))W) = u((f(z), 0)W) = j_1 \circ f(z) = g_2 \circ g(z)$$

Finally, u is unique, because if we have another homomorphism $v : X \oplus_Z Y \to C$ such that $v \circ i_2 \circ g = j_2 \circ g$ and $j_1 \circ f = v \circ i_1 \circ f$, we must have $u = v$. □

Proposition 0.16 (Exercise 52b)**.** *In the category of abelian groups, the push-out of an injective homomorphism is injective.*

Proof. Let f, g be objects in $\mathbf{Ab}^Z$ and let i_2 be the push-out of f by g, so we have $i_2 \circ g = i_1 \circ f$. Suppose that f is injective. We need to show that i_2 is injective. Because f is injective, it has trivial kernel. We compute

$$\ker i_2 = \{y \in Y : i_2(y) = W\} = \{y \in Y : (0, y) \in W\} = \{y \in Y : (0, y) = (f(z), -g(z))\}$$

Since $\ker f$ is trivial, this implies that $z = 0$ so $-g(z) = 0$ so $\ker i_2$ is trivial. Thus i_2 is injective. □